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Optical shock waves in media with quadratic nonlinearity

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We report the existence of optical shock waves in quadratic media with complex dispersion. By using a truncated Painleve method it is analytically shown that the second harmonic and fundamental fields can propagate into the unexcited region as a mutually trapped two-field shock wave. The conditions for the existence of this solution are determined. [S1063-651X(98)51410-9]

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An important tendency in contemporary studies on nonlinear optics is related to quadratic or $\chi^{(2)}$ nonlinearities which are believed to possess a high potential in all-optical signal processing, amplification, transistor action, generation of nonlinear phase shifts, pulse compression, etc. (see, e.g., [1,2] and references therein).

Many realistic physical systems exhibit inherent losses and/or gain that cannot be neglected in describing the wave dynamics (propagation and interaction). These dissipative systems can be frequently described by partial differential equations with complex coefficients. Prominent examples are the Ginzburg-Landau equation (GLE) and its different modifications. Different types of solutions to the GLE, such as solitonlike including algebraic ones, shocklike, sources, sinks, periodic, etc. have been demonstrated and their properties were analyzed systematically [3–9]. These studies exclusively dealt with cubic and quintic rather than quadratic media. To date solutions in nonconservative systems cover only the former case, being merely a limiting case of much richer $\chi^{(2)}$ phenomena. The latter have a wide significance beyond optics as well. Rapid progress in material sciences permits one to manufacture molecular crystals with the new properties. For instance, dynamics of excitations in organic crystalline superlattices that are constructed from molecules demonstrating Fermi-resonance is described by equations equivalent to those in quadratic optical media [10].

In this Rapid Communication we will show that a shock-type solution previously known in cubic media exists in the case of $\chi^{(2)}$ nonlinearity, too. Here, two coupled fields of the fundamental wave (FW) and the second harmonic (SH) are involved. We derive an analytical solution and define the criteria of its existence.

The system of equations describing wave propagation in a dispersive quadratically nonlinear medium has the form

$$i(A_x+\delta A_t)+(k_1+i\gamma_1)A+D_1A_{tt}+2\Gamma A^*B=0, \quad (1a)$$

$$i(B_x-\delta B_t)+(k_2+i\gamma_2)B+D_2B_{tt}+\Gamma A^2=0, \quad (1b)$$

where x is the propagation distance, t is the retarded time, A and B are normalized envelopes of the first and the second harmonics, $\gamma_{1,2}$ are linear gain or loss coefficients, $D_{1,2}$ are dispersion coefficients, $q=k_2-2k_1$ is the phase mismatch, $k_{1,2}$ are the wave numbers at two frequencies, δ is the walk-off parameter, and Γ is the nonlinear coefficient.

In what follows we assume that $D_{1,2}=D'_{1,2}+iD''_{1,2}$ are complex valued. For example, Eqs. (1) with complex $D_{1,2}$ describe the mean fields evolution in the system with fluctuating group velocities $\nu_{1,2}^{-1}=k'_{1,2}=(\nu_{1,2}^{-1})_0+\eta_{1,2}$, where parameters $\eta_{1,2}(x)$ are the fluctuations of harmonic's reciprocal group velocities. In the case when $\eta_{1,2}(x)$ are δ correlated random Gaussian processes $\langle\eta_{1,2}(x_1)\eta_{1,2}(x_2)\rangle=2\sigma_{1,2}^2\delta(x_1-x_2)$, here $\langle\ldots\rangle$ stands for the statistical averaging, application of the Furutsu-Novikov formula [11] to the averaged governing equations leads to the expression $D''_{1,2}=-\sigma_{1,2}^2$. Other spectacular examples of the systems with complex D are those with spectral filtering or bandwidth-limited amplification [9]. For real dispersion and vanishing linear losses (gain) Eqs. (1) simplify to the well-studied limit. Different types of localized solutions to this system, namely, bright, dark, and semidark have been discussed lately in the literature [12,13]. However, the existence of the solutions to system (1) with complex coefficients has not been reported to the time.

To find the solutions to system (1) we apply the so-called truncated Painleve expansion method [14]. An application of this method to GLE and to quartic nonlinear Schrödinger equation (NLSE) can be found elsewhere [15]. The Painleve approach is based on the assumption that some of nonlinear equations possess the formal solutions of the form

$$u(x,t)=\Phi^{\alpha}\sum_{j=0}^{\infty}u_j(x,t)\Phi^j, \quad (2)$$

where $\alpha<0$ is the leading order power coefficient, $u_j(x,t)$ are expansion coefficients that are analytic in the neighborhood of the noncharacteristic singular manifold $\Phi(x,t)=0$ [16]. The Painleve analysis allows one to investigate the integrability of differential equations, to systematically construct the Lax pair and the Bäcklund transformation as well as the Hirota bilinear representation. In recent papers [17–20] this approach was applied to study the coupled system of Maxwell-Bloch and Hirota equations [17], higher-order nonlinear Schrödinger equation that includes third-order dispersion, self-steepening and self-frequency shifting via stimulated Raman scattering terms [18,19], and coupled systems of equations describing the pulse propagation in quadratically nonlinear media accounting for third-order dispersion and self-steepening effects [20].

As shown in [14] the information about the Lax pair and the Bäcklund transformation for integrable system can be deduced by using the expansion (2) truncated at the term with $j=-\alpha$. Moreover, by substituting the truncated Laurent series (2) into the governing equation one can find particular solutions both in integrable and in nonintegrable cases [14,15]. Here we take advantage of this approach.

We suppose that the solution to Eqs. (1) has a singularity at complex values of t and x along the curve $\Phi(x,t)=0$. For real x and t, the function $\Phi(x,t)$ is real, too. Looking for the leading order power coefficients of the two fields we substitute

$$A=a\Phi^{\alpha}, \quad B=b\Phi^{\beta} \tag{3}$$

into Eq. (1), and upon balancing the dominant terms and collecting the coefficients of the leading power of $\Phi(x,t)$ we can conclude that

(i) The parameters α, β are complex valued,

$$\alpha=-2+i\varepsilon, \quad \beta=-2+2i\varepsilon; \tag{4}$$

(ii) The parameter ε and the coefficients $a(x,t)$ and $b(x,t)$ are determined by the following equations:

$$\alpha(\alpha-1)D_1 a\Phi_t^2+2\Gamma a^* b=0, \tag{5a}$$

$$\beta(\beta-1)D_2 b\Phi_t^2+\Gamma a^2=0. \tag{5b}$$

Then substituting the series (2) truncated at $j=2$ and collecting the coefficients of different power of $\Phi(x,t)$ we obtain a rather complicated set of equations for $\Phi(x,t)$, a_j, b_j $(j=0,1,2)$. However, this system of equations permits a special solution with $a_j=b_j=0$ for $j=1,2$. Below we analyze this case.

The procedure described above leads to the following set of equations:

$$i(\Phi_x+\delta\Phi_t)a+D_1(2a_t\Phi_t+a\Phi_{tt})=0, \tag{6a}$$

$$i(\Phi_x-\delta\Phi_t)b+D_2(2b_t\Phi_t+b\Phi_{tt})=0, \tag{6b}$$

$$i(a_x+\delta a_t)+(k_1+i\gamma_1)a+D_1 a_{tt}=0, \tag{7a}$$

$$i(b_x-\delta b_t)+(k_2+i\gamma_2)b+D_2 b_{tt}=0, \tag{7b}$$

which together with Eq. (5) represent an overdetermined system to be solved. Similar to Eq. (3), in Eqs. (6) and (7) we have omitted the subscript 0 for a and b.

The analysis of Eq. (5) shows that the parameter ε is determined by the fourth-order algebraic equation

$$\begin{aligned}&2(D_1'D_2''+D_2'D_1'')(\varepsilon^4-20\varepsilon^2+9)\\&+15(D_1'D_2'-D_1''D_2'')(\varepsilon^3-3\varepsilon)=0,\end{aligned} \tag{8}$$

whereas for the functions a and b we obtain the following expressions:

$$\begin{aligned}a&=a_0\Phi_t^2\exp[i\varphi_1+i\varphi(x,t)],\\ b&=b_0\Phi_t^2\exp[i\varphi_2+2i\varphi(x,t)],\end{aligned} \tag{9}$$

$$\begin{aligned}a_0^2=&[2(D_1'D_2'-D_1''D_2'')(\varepsilon^4-20\varepsilon^2+9)\\&+15(D_1'D_2''+D_1''D_2')(3\varepsilon-\varepsilon^3)]/\Gamma^2,\\ b_0^2=&|D_1|^2(\varepsilon^4+13\varepsilon^2+36)/4\Gamma^2,\end{aligned} \tag{10}$$

$$\tan(\varphi_2-2\varphi_1)=\frac{D_1''(6-\varepsilon^2)-5D_1'\varepsilon}{D_1'(6-\varepsilon^2)+5D_1''\varepsilon}. \tag{11}$$

Thus, Eqs. (10) and (11) determine the amplitudes and the constant part of the relative phase difference of the two fields.

Now to find functions $\varphi(x,t)$ and $\Phi(x,t)$ we have four Eqs. (6) and (7). Substitution of Eq. (9) into Eq. (6) leads (i) to the expression for the phase

$$\begin{aligned}\varphi(x,t)&=\mu\ \ln\Phi_t+\varphi_0(x),\\ \mu&=5D_1'/2D_1'',\end{aligned} \tag{12}$$

where $\varphi_0(x)$ is an arbitrary function that will be determined later; (ii) to the condition of compatibility of Eqs. (6a) and (6b),

$$2D_1'/D_1''=D_2'/D_2''; \tag{13}$$

and (iii) to two equations for $\Phi(x,t)$,

$$D_1''(\Phi_x+\delta\Phi_t)+5|D_1|^2\Phi_{tt}=0, \tag{14a}$$

$$D_2''(\Phi_x-\delta\Phi_t)+5|D_2|^2\Phi_{tt}=0. \tag{14b}$$

Before analyzing the solutions of the system (14) we consider the equations that are consequences of the system (7). Substituting Eq. (9) into Eq. (7) we arrive at

$$\begin{aligned}&(k_1+i\gamma_1-\varphi_{0_x})\Phi_t^2+D_1(1+i\mu)(2+i\mu)\Phi_{tt}^2\\&-2D_1(2+i\mu)^2\Phi_t\Phi_{ttt}=0,\end{aligned} \tag{15a}$$

$$\begin{aligned}&(k_2+i\gamma_2-2\varphi_{0_x})\Phi_t^2+2D_2(1+i\mu)(1+2i\mu)\Phi_{tt}^2\\&-8D_1(1+i\mu)^2\Phi_t\Phi_{ttt}=0.\end{aligned} \tag{15b}$$

Introducing the new function $F=\Phi_{tt}/\Phi_t$ it is easy to show that Eq. (15a) has a solution of the form

$$\Phi(x,t)=c_1(x)+c_2(x)e^{Ft}, \tag{16}$$

where F is a constant determined by the imaginary part of Eq. (15a),

$$F^2=4\gamma_1 D_1''/(25|D_1|^2-D_1''^2)>0, \tag{17}$$

whereas the real part of Eq. (15a) gives

$$\begin{aligned}&\varphi=Q_1 x,\\ &Q_1=k_1+\gamma_1 D_1'(25|D_1|^2+D_1''^2)/D_1''(25|D_1|^2-D_1''^2).\end{aligned} \tag{18}$$

The compatibility condition of Eqs. (15a) and (15b) yields

$$\gamma_1 D_1''(25|D_2|^2 - D_2''^2) = \gamma_2 D_2''(25|D_1|^2 - D_1''^2), \quad (19)$$

$$k_2 - 2k_1 = \frac{D_2' F^2}{4D_2''}\left[25\left(\frac{|D_2|^2}{D_2''} - \frac{|D_1|^2}{D_1''}\right) + D_2'' - D_1''\right], \quad (20)$$

where relation (13) was used.

Now we come back to Eq. (14). Substituting Eq. (16) into Eq. (14) we find that the common solution to Eqs. (14) and (15) has the form

$$\Phi(x,t) = c_0 + c e^{F(t - x/V)}, \quad (21)$$

where c_0 and c are constants, which can be equalized by a proper choice of the reference points x_0 and t_0, and

$$V^{-1} = (5F|D_1|^2 + \delta)/D_1'' = (5F|D_2|^2 - \delta)/D_2''. \quad (22)$$

The second equality in Eq. (22) arising from the compatibility condition for Eqs. (14a) and (14b) shows that the solution exists if the condition

$$\delta = 5F(D_1''|D_2|^2 - D_2''|D_1|^2)/(D_2'' + D_1'') \quad (23)$$

holds. By using Eq. (23) the expression for velocity (22) can be rewritten in the form

$$V^{-1} = 5F(|D_2|^2 + |D_1|^2)/(D_2'' + D_1''). \quad (24)$$

For zero walk-off instead of Eq. (23) an additional relation for coefficients $D_{1,2}$,

$$D_1''|D_2|^2 = D_2''|D_1|^2 \quad (25)$$

emerges. Eventually the solution of the system (1) is given by

$$A = A_0[1 + e^{-F(t - x/V)}]^{-2 + i\varepsilon} e^{i(\Omega t - Qx) + i\varphi_1},$$

$$B = B_0[1 + e^{-F(t - x/V)}]^{-2 + 2i\varepsilon} e^{2i(\Omega t - Qx) + i\varphi_2}, \quad (26)$$

where $A_0 = a_0 F^2$, $B_0 = b_0 F^2$, $\Omega = (\mu + \varepsilon)F$, $Q = Q_1 + (\mu + \varepsilon)F/V$, parameters a_0, b_0, F, Q_1, and V are given by expressions (10), (17), (18), and (22), respectively. Thus Eq. (26) represents the solution of system (1) with fixed values of velocity and amplitudes of both harmonics. It is clear that for $F > 0$ in the limit $t - x/V \to -\infty$ both amplitudes tend to zero, whereas in the limit $t - x/V \to \infty$ we obtain nonzero excitations, $A \to A_0$, $B \to B_0$. The solution (26) exists if the parameters of system (1) satisfy the relations (13), (19), (20), and (23). Note that transformations $t \to t - t_0$, $x \to x - x_0$, and $\varphi_1 \to \varphi_1 + \varphi_0$, $\varphi_2 \to \varphi_2 + 2\varphi_0$ in Eq. (26) leads to the solution of Eq. (1) as well.

Now two comments are in order. As follows from Eqs. (17) and (19) the inequalities $\gamma_{1,2} D_{1,2}'' > 0$ have to be satisfied. In other words, $\mathrm{sgn}(\gamma_{1,2}) = \mathrm{sgn}(D_{1,2}'')$, which is a physically obvious condition because the system has to exhibit a balance between gain and losses to sustain a stationary solution.

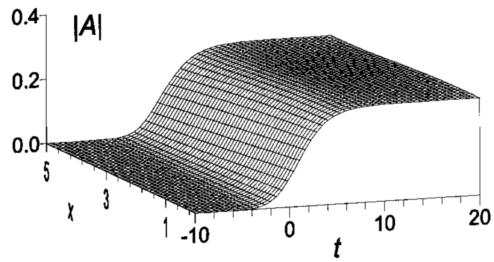

FIG. 1. Evolution of the shock wave. Amplitude of the fundamental wave A is shown, while the SH field demonstrates a similar behavior.

The second comment is that Eq. (8) in general can have up to four real solutions. If they are consistent with Eq. (10) one set of system parameters admits several coexisting shocklike solutions.

Accounting for the relation (13), Eq. (8) can be rewritten as

$$2d(\varepsilon^4 - 20\varepsilon^2 + 9) + 5(2d - 1)(\varepsilon^3 - 3\varepsilon) = 0, \quad (27)$$

where $d = D_1'/D_1''$. For instance, in the case $d = 0.5$ Eq. (27) is reduced to a biquadratic equation and has four real solutions: $\varepsilon_i = \pm[10 \pm \sqrt{91}]^{1/2}$. Analysis of Eq. (10) shows that $a_{0i}^2 = 10 D_1'' D_2''(3\varepsilon_i - \varepsilon_i^3)/\Gamma^2 > 0$ and therefore there are two pairs of solutions with $\varepsilon_1 = -[10 + \sqrt{91}]^{1/2}$, $\varepsilon_2 = [10 - \sqrt{91}]^{1/2}$ and $\varepsilon_3 = -[10 - \sqrt{91}]^{1/2}$, $\varepsilon_4 = [10 + \sqrt{91}]^{1/2}$. For the former case we have $\mathrm{sgn}(D_1'') = \mathrm{sgn}(D_2'')$, while for the latter $\mathrm{sgn}(D_1'') = -\mathrm{sgn}(D_2'')$.

Another particular case we consider here is the dispersionless limit $d = 0$. Then solutions to Eq. (27) are $\varepsilon_1 = 0$, $\varepsilon_{2,3} = \pm\sqrt{3}$. For $\varepsilon_1 = 0$ we have $a_{01}^2 = -18 D_1'' D_2''/\Gamma^2$, i.e., a solution exists provided that $\mathrm{sgn}(D_1'') = -\mathrm{sgn}(D_2'')$, while for $\varepsilon_{2,3}^2 = 3$, $a_{02}^2 = a_{03}^2 = 84 D_1'' D_2''/\Gamma^2$, and two signs of ε correspond to the coexistence of the two complex conjugate solutions. Along with these relations, Eqs. (17)–(20) lead to $F^2 = \gamma_1/6D_1'' = \gamma_2/6D_2''$, $Q_1 = k_1$, $k_2 = 2k_1$. The last equality means that in the dispersionless limit solutions exist at zero mismatch.

We have confirmed the results obtained by direct numerical simulations of the Eq. (1). In Fig. 1 the dynamics of the shock wave is shown for the system parameters $D_1' = D_2' = 0$, $D_1'' = D_2'' = 0.1$, $\delta = 0$, $\gamma_1 = \gamma_2 = 0.2$, $\varepsilon = \sqrt{3}$. A rather stable propagation of the shock wave with amplitudes, relative phase, velocity, and shape described by the formulas (10), (11), (24), and (26), respectively, is clearly seen.

In conclusion, we have analytically found the profile of a shock front formed by the interaction of the fundamental and the second harmonic waves in nonconservative quadratic media. Due to mutual trapping the FW and the SH compose a bound state and move with a common velocity even in the case of nonzero walk-off. It is shown that for a set of parameters characterizing dispersive, nonlinear, dissipative, and amplifying properties of the system several shocklike waves can exist. An important problem that remains beyond the scope of this Rapid Communication is the stability of the found solutions. Numerical calculations showed that at least for particular set of the system parameters shock waves propagate rather stable. A detailed analysis of the stability of the shock waves, in particular, modulational instability of the plane waves which are the asymptotic states of the solutions (26), will be considered elsewhere. Such shock waves can be observed experimentally if long pulses with sharp fronts are launched in a dissipative quadratic medium.

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[1] G. I. Stegeman, D. J. Hagan, and L. Torner, Opt. Quantum Electron. **28**, 1691 (1996).
[2] A. Kobyakov and F. Lederer, Phys. Rev. A **54**, 3455 (1996).
[3] P. Kolodner, Phys. Rev. A **46**, 6431 (1992); **46**, 6452 (1992).
[4] R. J. Deissler and H. R. Brand, Phys. Rev. Lett. **72**, 478 (1994).
[5] N. Akhmediev and V. V. Afanasjev, Phys. Rev. Lett. **75**, 2320 (1995).
[6] N. Akhmediev and A. Ankiewicz, Solitons, Nonlinear Pulses and Beams (Chapman & Hall, London, 1997).
[7] B. Malomed and A. Nepomnyashchy, Phys. Rev. A **42**, 6009 (1990).
[8] D. Cai, A. R. Bishop, N. Cronbech-Jensen, and B. Malomed, Phys. Rev. Lett. **78**, 223 (1997).
[9] G. P. Agrawal, Nonlinear Fiber Optics (Academic Press, New York, 1995).
[10] V. M. Agranovich, O. A. Dubovsky, and A. M. Kamchatnov, J. Phys. Chem. **98**, 13 607 (1994); Chem. Phys. **198**, 245 (1995).
[11] V. I. Klyatskin, *Stochastic Equations and Waves in Randomly Inhomogeneous Media* (Nauka, Moscow, 1980; Editions de Physique, Besancon, 1985).
[12] C. R. Menyuk, R. Schiek, and L. Torner, J. Opt. Soc. Am. B **11**, 2434 (1994).
[13] A. V. Buryak and Yu. S. Kivshar, Phys. Rev. A **51**, 41 (1995).
[14] A. C. Newell, M. Tabor, and Y. B. Zeng, Physica D **29**, 1 (1987).
[15] F. Cariello and M. Tabor, Physica D **39**, 77 (1989).
[16] J. Weiss, M. Tabor, and G. Carnevale, J. Math. Phys. **24**, 522 (1983).
[17] K. Porsezian and K. Nakeeran, Phys. Rev. Lett. **74**, 2941 (1995).
[18] K. Porsezian and K. Nakeeran, Phys. Rev. Lett. **76**, 3955 (1996).
[19] M. Gedalin, T. C. Scott, and Y. B. Band, Phys. Rev. Lett. **78**, 448 (1997).
[20] D. Mihalache, L. C. Crasovan, and N. C. Panoiu, J. Phys. A **30**, 5855 (1997).